MATHEMATICS

ON INTEGRATION THEORY

BY

T. P. SRINIVASAN

(Communicated by Prof. J. F. KOKSMA at the meeting of November 28, 1959)

The classical measure theoretic approaches and Stone's linear functional approach to the theory of Integration are well-known. The classical approaches are mainly the "simple function approach" and the "ordinate set approach". In either of them one starts with a measure (Λ, λ) in a set X; in the former, one first defines the integral for simple functions in X and uses their properties to define the integral for arbitrary measurable functions; in the latter, one constructs the product measure of (Λ, λ) with the Lebesgue measure in R_1^+ (the space of non-negative reals) and defines the integrability and the integral in terms of the measurability and the measure of ordinate sets of functions. In Stone's approach on the other hand, one starts from an "elementary integral" on a vector lattice of functions which is defined by essentially axiomatising the properties of integrals of simple functions with respect to a measure and then extends this to a wider class, the class of "integrable" and "summable" functions, without any reference to a measure.

ZAANEN has in his recent book [3] given a variant of Stone's approach combining it with the ordinate set approach. He starts with Stone's assumptions but introduces the integral in terms of measures of ordinate sets with respect to a certain measure $(\bar{\Gamma}, \gamma)$ that he constructs in the product space $X \times R_1^+$. But in the case when the resulting integral is a "Lebesgue-Stieltjes" integral (i.e. an integral in the conventional sense), the fact that it is so, is proved only at the end. That is because, it is not known in the beginning, that in this case the measure $(\bar{\Gamma}, \gamma)$ is a genuine product measure of some measure in X and the Lebesgue measure in R_1^+. We show in this paper directly, using only the standard properties of measures, that $(\bar{\Gamma}, \gamma)$ is indeed the associated product measure of some measure in X, and thereby reduce at the outset Zaanen's approach to the ordinate set approach. We also rederive the equivalence of the simple function and the ordinate set approaches and thereby conclude the equivalence of Zaanen's approach with the classical simple functional approach, our proof of equivalence differing from the known proofs in that we do not need anywhere in the course of the proof any of the properties of integrable or measurable functions beyond their definitions. The proof uses only the standard properties of measure.

Incidentally some consistency theorems for which separate proofs were needed in [3] follow as immediate corollaries in our method of proof.

2. *Notation* and *preliminary results:*

With regard to set unions we use the symbol $\sum$ to denote disjoint unions while we preserve the usual symbol $\bigcup$ to denote unions, not necessarily disjoint.

We need our measures only to be defined on semi-rings of subsets of X; where by a *semi-ring* we mean a class Λ of subsets such that

$$A, B \in \Lambda \Rightarrow \begin{cases} A \cap B \in \Lambda, \\ A - B = \sum_{i=1}^{n} C_i, C_i \in \Lambda \text{ for some integral } n. \end{cases}$$

Given a measure (Λ, λ), let Λ_1 denote the class of all sets $E \in \Lambda$ with $\lambda(E) < \infty$, and λ_1 the restriction of λ to Λ_1, written in symbols as $\lambda_1 = \lambda | \Lambda_1$. Then (Λ_1, λ_1) is a finite measure, and we call this the *finite part* of (Λ, λ).

Let λ^* be the *outer measure* induced by (Λ, λ) and Λ^* the class of λ^*-measurable sets. We call the measure (Λ^*, λ) the *extended measure space* of (Λ, λ), where we continue to denote by λ the restriction of λ^* to Λ^*.

We shall repeatedly use the following lemma:

Lemma 1

The outer measures λ_i^* generated by two measures (Λ_i, λ_i), $i = 1, 2$ are the same if and only if $\lambda_1^* | \Lambda_2 = \lambda_2$ and $\lambda^*_2 | \Lambda_1 = \lambda_1$.
For proof we refer to [3], p. 23, Theorem 2.

A *Simple function* for (Λ, λ) is a function which assumes only finitely many distinct values, each on a set in Λ, i.e. a function f of the type

$$f = \sum \alpha_i \chi_{Ei}$$

where the α_i's are real numbers, E_i's are sets in Λ, χ_{Ei} is the *characteristic function* of E_i and the sum on the right is finite. (We allow α_i to be ∞ provided $\lambda(E_i) = 0$). For such f's, the integral $\mathrm{I}(f)$ is defined as the sum:

$$I(f) = \sum \alpha_i \lambda(E_i),$$

with the standing convention that $0 . \infty = 0$.

The simple function f above is a *step function* for (Λ, λ) if $\lambda(E_i) < \infty$ for all i. For step functions f, the values $I(f)$ are necessarily finite.

An *elementary integral* is a pair (L, I) where

i. L is a vector lattice of real valued functions; i.e.:

   $f, g \in L;\ a, b \text{ real} \Rightarrow af + bg,\ f \cup g,\ f \cap g \in L$

ii. I is a non-negative linear functional on L, which is further continuous for pointwise monotone increasing limits; i.e.:

$I(af+bg)=aI(f)+bI(g)$

$f \geqslant g \Rightarrow I(f) \geqslant I(g)$; in particular, $f \geqslant 0 \Rightarrow I(f) \geqslant 0$,

$f_n, f \in L^+$, $f_n \uparrow f \Rightarrow I(f_n) \uparrow I(f)$, where $L^+ = [f | f \in L,\ f \geqslant 0]$ and finally,

iii. $f \in L \Rightarrow f \cap 1 \in L$.

The following result is standard and is easily proved.

Lemma 2

If S is the class of all simple functions for a measure (Λ, λ), and I denotes their integral, and if J is the class of all step functions, (S, I) is an elementary integral and (J, I) is a finite elementary integral.

We denote by Δ the semi-ring of half open intervals $[\alpha, \beta)$ in $R_1{}^+$ and by δ the length function. Then Δ is a semi-ring and (Δ, δ) is a measure. Given a measure (Λ, λ), we denote by $\bar{\Lambda}$ the class $\Lambda \times \Delta$ of rectangles $A \times B$, where $A \in \Lambda$ and $B \in \Delta$; and by $\bar{\lambda}$ the set function $\lambda \times \delta$ on $\bar{\Lambda}$, defined by $\bar{\lambda}(A \times B) = \lambda(A)\delta(B)$. We have the following:

Theorem 1

$$(\bar{\Lambda}, \bar{\lambda}) \text{ is a measure in } X \times R_1{}^+.$$

The verification that $\bar{\Lambda}$ is a semi-ring is straightforward. Also $\bar{\lambda}$ is easily seen to be monotone and non-negative. The non-trivial part is the countable additivity of $\bar{\lambda}$; and this follows from the continuity under monotone limits of the elementary integral on simple functions for (Λ, λ). In fact, if $A \times [\alpha, \beta) = \sum_1^\infty (A_i \times [\alpha_i, \beta_i))$, then $(\beta - \alpha)\chi_A = \sum_1^\infty (\beta_i - \alpha_i)\chi_{A_i}$. Now if $f_n = \sum_1^n (\beta_i - \alpha_i)\chi_{A_i}$ and $f = (\beta - \alpha)\chi_A$, clearly, f_n, f are simple functions and $f_n \uparrow f$, so that $I(f_n) \uparrow I(f)$. The desired conclusion follows.

The measure $(\bar{\Lambda}, \bar{\lambda})$ is defined to be the *associated product measure* of (Λ, λ) and its extended measure space $(\bar{\Lambda}^*, \bar{\lambda})$, the *associated product measure space.*

For any set $S \subset X \times R_1{}^+$ and real number a, we denote by S_a the section of S at a; i.e. $S_a = [x / x \in X,\ (x, a) \in S]$.

We need the following regarding the associated product measure.

Lemma 3

i. $S \in \bar{\Lambda}^* \Rightarrow S_a \in \Lambda^*$ for almost all a (i.e. except for a set of a's of Lebesgue measure zero).

ii. $E \times [\alpha, \beta) \in \bar{\Lambda}^*$ for some $\beta > \alpha \geqslant 0 \Rightarrow E \in \Lambda^*$.

iii. $E \in \Lambda^* \Rightarrow E \times [\alpha, \beta) \in \bar{\Lambda}^*$ for every $\beta > \alpha \geqslant 0$, and then $\bar{\lambda}(E \times [\alpha, \beta)) = (\beta - \alpha)\lambda(E)$.

Proof of i):

If S is of the form $S = \sum_1^\infty (A_i \times [\alpha_i, \beta_i))$, $A_i \in \Lambda$, it is obvious that S_a is a countable union of the A_i's and therefore belongs to Λ^* for every real a. Further, if for any p, $0 \leqslant p < \infty$,

$$_pS = [a/\lambda^*(S_a) > p],$$

we shall show that in this case $_pS \in \Lambda^*$, and $p.\delta(_pS) \leqslant \bar{\lambda}(S)$.

It suffices to prove this when S is a finite union; in the alternative case, if S_n denotes the truncated sum $\sum_1^N (A_i \times [\alpha_i, \beta_i))$, then $S_N \uparrow S$, $(S_N)_a \uparrow S_a$ and $_pS_N \uparrow {_pS}$; so that the validity of the result for $_pS_N$, for all N, will trivially imply that for $_pS$. Let then $S = \sum_1^R (A_i \times [\alpha_i, \beta_i))$. Then, clearly, $\chi_{Sa} = \sum_1^R \chi_{[\alpha i, \beta i)}^{(a)} \chi_{Ai}$ is a simple function, and $\lambda(S_a) = I(\chi_{Sa}) = \sum_1^R \lambda(A_i) \chi_{[\alpha i, \beta i)}^{(a)}$ for every real a. Now the function $f = \sum_1^R \lambda(A_i) \chi_{[\alpha i, \beta i)}$ is a simple function for the Lebesgue measure, and $_pS$ is precisely the set $E = [f > p]$. Then $f \geqslant f\chi_E \geqslant p\chi_E$ and hence, by elementary properties of simple functions, $I(f) \geqslant I(p\chi_E) = p.\delta(E)$. Clearly $I(f) = \bar{\lambda}(S)$ and $\delta(E) = \delta(_pS)$; and the result follows.

If S is a $\bar{\lambda}$-null set, almost all S_a's are λ-null and therefore belong to Λ^*. For, given integres n, r, there exists a $T = \sum_1^\infty (A_i \times [\alpha_i, \beta_i))$, $A_i \in \Lambda$, such that $T \supset S$ and $\bar{\lambda}(T) < 1/n^r$. Then $_pS \subset {_pT}$ for every p; and taking $p = 1/n$ in the above, we have

$$\frac{1}{n}\delta(_{1/n}S) \leqslant \frac{1}{n}\delta(_{1/n}T) \leqslant \bar{\lambda}(T) < \frac{1}{n^r},$$

so that $\delta(_{1/n}S) = 1/n^{r-1}$. For a fixed n, letting $r \to \infty$, this implies that $\delta(_{1/n}S) = 0$. In other words, $\delta[a/\lambda^*(S_a) > 1/n] = 0$ for all n and we conclude that $\delta[a|\lambda^*(S_a) > 0] = 0$.

If Θ denotes the class of all sets S for which (i) is true, Θ is a σ-ring in the first place, and it contains all sets in $\bar{\Lambda}$ and all $\bar{\lambda}$-null sets. Θ therefore contains all sets of finite $\bar{\lambda}$-measure. As a result, if $S \in \bar{\Lambda}^*$ and $\bar{\lambda}(S) = \infty$, then $S_a \cap E \in \Lambda^*$ for all sets E of finite measure in Λ; and this in turn implies that $S_a \in \Lambda^*$. It follows that $\Theta \supset \bar{\Lambda}^*$, completing the proof of (i).

Proof of ii) and iii)

If $\bar{E} = E \times [\alpha, \beta) \in \bar{\Lambda}^*$, then $E = (\bar{E})_a$ for any $a \in [\alpha, \beta)$, and therefore E belongs to Λ^*, by (i). Conversely, let α and β be fixed ($\beta > \alpha \geqslant 0$). Then the class $\{E | E \times [\alpha, \beta) \in \bar{\Lambda}^*\}$ is easily seen to be a σ-ring containing Λ and all λ-null sets, and therefore also all sets of finite λ-measure. If $E \in \Lambda^*$ and $\lambda(E) = \infty$, then $(E \times [\alpha, \beta)) \cap S \in \bar{\Lambda}^*$ for every set $S \in \bar{\Lambda}$ of finite measure, and then $E \times [\alpha, \beta) \in \bar{\Lambda}^*$ itself.

It only remains to show that $\bar{\lambda}(\bar{E}) = (\beta - \alpha)\lambda(E)$. It is easy to see that $\bar{\lambda}(\bar{E}) \leqslant (\beta - \alpha)\lambda(E)$. To prove the reverse inequality, we can assume that $\bar{\lambda}(E) < \infty$ and $\lambda(E) > 0$. There exists $S = \sum_1^\infty A_i \times B_i \supset \bar{E}$, $A_i \times B_i \in \bar{\Lambda}$, such that $\sum_1^\infty \bar{\lambda}(A_i \times B_i) < \bar{\lambda}(\bar{E}) + \varepsilon$. For any a, $\bar{E}_a \subset S_a$; and for $a \in [\alpha, \beta)$,

$\bar{E}_a = E$. Then for $0 \leqslant p < \lambda(E)$, we have ${}_pS \supset [\alpha, \beta)$, and consequently $p(\beta-\alpha) \leqslant p\delta({}_pS) \leqslant \sum_i \bar{\lambda}(A_i \times B_i) < \bar{\lambda}(\bar{E}) + \varepsilon$. Since $0 \leqslant p < \lambda(E)$ is arbitrary, it follows that $(\beta-\alpha)\lambda(E) \leqslant \bar{\lambda}(\bar{E})$.

Using lemma 3 we prove the following theorems.

Theorem 2

The associated product measure space of (Λ, λ) is also the associated product measure space of (Λ^*, λ).

Proof:

We have only to show that the outer measures induced by the measures $(\Lambda \times \Delta, \lambda \times \delta)$ and $(\Lambda^* \times \Delta, \lambda \times \delta)$ are equal. Let λ_1, λ_2 denote the restrictions: $\lambda \times \delta | \Lambda \times \Delta$, and $\lambda \times \delta | \Lambda^* \times \Delta$. Then by lemma 1 it suffices to show that $\lambda_2{}^* | \Lambda \times \Delta = \lambda_1$ and $\lambda^*{}_1 | \Lambda^* \times \Delta = \lambda_2$. The first part is trivial since on $\Lambda \times \Delta$, $\lambda_2{}^* = \lambda_2 = \lambda_1$; the other part follows from lemma 3, (iii).

Theorem 3

If the associated product measure spaces of two measures (Λ_i, λ_i), $i = 1, 2$, are identical, then so are their extended measure spaces; in symbols: if $(\bar{\Lambda}_1{}^*, \bar{\lambda}_1) \equiv (\bar{\Lambda}_2{}^*, \bar{\lambda}_2)$, then $(\Lambda_1{}^*, \lambda_1) \equiv (\Lambda_2{}^*, \lambda_2)$.

Proof:

$$\begin{aligned} E \in \Lambda_1{}^* &\Rightarrow \bar{E} = E \times [0, 1) \in \bar{\Lambda}_1{}^*, \text{ and } \bar{\lambda}_1(\bar{E}) = \lambda_1(E) \text{ (by Lemma 3 (iii))} \\ &\Rightarrow \bar{E} \in \bar{\Lambda}_2{}^* \text{ and } \bar{\lambda}_2(\bar{E}) = \bar{\lambda}_1(\bar{E}) = \lambda_1(E) \\ &\Rightarrow E \in \Lambda_2{}^* \text{ and } \lambda_2(E) = \bar{\lambda}_2(\bar{E}) = \lambda_1(E) \quad \text{(by lemma 3 (ii))}; \end{aligned}$$

by symmetry $E \in \Lambda_2{}^* \Rightarrow E \in \Lambda_1{}^*$ and $\lambda_1(E) = \lambda_2(E)$; and the proof is complete.

For any set $E \subset X$ we shall continue to denote by $\bar{E}$ the set $E \times [0, 1)$ in $X \times R_1{}^+$. For $S \subset X \times R_1{}^+$ and $a \in R_1{}^+$ we define aS by: $aS = [(x, ay) | (x, y) \in S]$. In particular, for $E \subset X$, $a\bar{E} = E \times [0, a)$.

3. *The classical approaches*

We assume as given a measure (Λ, λ). The simple functions and measurable functions we consider below are all with respect to the extended measure space (Λ^*, λ). For non-negative functions f on X we introduce the *ordinate set* Ω_f by

$$\Omega_f = [(x, y) | x \in X, y \in R_1{}^+, \ 0 \leqslant y < f(x)].$$

We observe that f is a measurable function if and only if Ω_f is a measurable set for the associated measure space $(\bar{\Lambda}^*, \bar{\lambda})$.

For, if f is measureable, the sets $E_a = [f > a] \in \Lambda^*$, for all a. In particular, if $\{a_n\}$ is a suitable countable dense subset of $R_1{}^+$, then $E_{a_n} \in \Lambda^*$, so that $a_n \bar{E}_{a_n} \in \bar{\Lambda}^*$ for all n, by lemma 3; and therefore also their union $\bigcup_n (a_n \bar{E}_{a_n})$ which is nothing but Ω_f.

Conversely, if $\Omega_f \in \bar{\Lambda}^*$, then almost all sections $(\Omega_f)_a$ belong to Λ^*, by lemma 3 (i); in particular, $(\Omega_f)_a \in \Lambda^*$ for a countable dense set of a's in $R_1{}^+$. Since $(\Omega_f)_a$ is just the set $[f > a]$, this implies that f is measurable.

We recall the classical definition of integrals.

Definition A_1

The integral on X of a non-negative measurable function f is defined by

$$\int f \, d\lambda = \lim_n I(f_n),$$

where $\{f_n\}$ is a non-decreasing sequence of non-negative simple functions converging to f, and $I(f_n)$ the integral of f_n, defined earlier.

This definition is both meaningful and unambiguous. Equivalently we have:

Definition A_2

The integral on X of a non-negative measurable function is the measure of the ordinate set Ω_f of f, in the associated product measure space $(\bar{\Lambda}^*, \bar{\lambda})$ of the measure (Λ, λ).

This definition is meaningful since $\Omega_f \in \bar{\Lambda}^*$ whenever f is measurable. It also agrees with the definition A_1. For, if f is non-negative simple, say, $f = \sum \alpha_i \chi_{Ei}$, where the E_i's belong to Λ^* and are disjoint, then

$$\Omega_f = \textstyle\sum_i \alpha_i \bar{E}_i, \text{ and } \bar{\lambda}(\Omega_f) = \sum_i \bar{\lambda}(\alpha_i \bar{E}_i) = \sum \alpha_i \lambda(E_i) = I(f).$$

If now f is any non-negative measurable function, and if the f_n's are as in definition A_1, then $\Omega_{f_n} \uparrow \Omega_f$, so that

$$\bar{\lambda}(\Omega_f) = \lim_n \bar{\lambda}(\Omega_{f_n}) = \lim_n I(f_n) = I(f).$$

The definition of integral is extended, and integrability and summability defined for arbitrary measurable functions, in the usual way, by decomposing f into its positive and negative parts.

4. *A Variant of Stone's approch – The main theorem*

Let (L, I) be a finite elementary integral in X, and let $L^+ = [f / f \in L, f \geqslant 0]$. Let Γ denote the class of all sets $\Omega_f - \Omega_g$ where $f, g \in L^+$, $f \geqslant g$ and define $\gamma(\Omega_f - \Omega_g) = I(f - g)$.

Lemma 4

$$(\Gamma, \gamma) \text{ is a measure in } X \times R_1{}^+.$$

For proof we refer to [3], p. 48. Theorem 2.

We call the measure (Γ, γ) the *associated product measure of* (L, I), and its extended measure space (Γ^*, γ) the *associated product measure space.*

Lemma 5

$$S \in \Gamma^* \Rightarrow aS \in \Gamma^*, \text{ and } \gamma(aS) = a.\gamma(S) \text{ for } a > 0.$$

For proof we refer to [3], p.49, Theorem 4.

We define a non-negative function f to be measurable for (L, I), or briefly I-measurable, if $\Omega_f \in \Gamma^*$; and then we define $I(f) = \gamma(\Omega_f)$. The concepts of I-measurability, integrability and summability, and the values $I(f)$ for arbitrary f are then defined in the usual way by considering the positive and negative parts of f. Trivially, all the functions in L are I-summable and I extends the given functional on L.

A set $E \subset X$ is said to be measurable if χ_E is I-measurable, and its measure $\lambda(E)$ is defined to be $I(\chi_E)$. If Λ is the class of all measurable subsets of X, it is easily seen that Λ is a σ-ring and (Λ, λ) is a measure. We call this the *induced measure* of (L, I).

We have the following main theorem:

Theorem 4 (cf. [3], p. 69, Theorem 7)

i) $(\bar{\Lambda}^*, \bar{\lambda}) = (\Gamma^*, \gamma)$

ii) The concepts of measurability, integrability, summability etc., and the values $I(f)$ for (L, I), coincide with the corresponding concepts for (Λ, λ).

iii) In particular, the functions in L are all summable for (Λ, λ) and $I(f) = \int f\, d\lambda$.

Proof

We need only prove (i); the rest follows. We shall equivalently show that $(\bar{\Lambda}, \bar{\lambda})$ and (Γ, γ) both generate the same outer measure. By lemma 1, it suffices to show that: (a) $\gamma^* | \bar{\Lambda} = \bar{\lambda}$; (b) $\bar{\lambda}^* | \Gamma = \gamma$.

Proof of (a):

$$S = E \times [\alpha, \beta) \in \bar{\Lambda} \Rightarrow E \in \Lambda \Rightarrow \bar{E} \in \Gamma^* \Rightarrow \alpha\bar{E},\ \beta\bar{E} \in \Gamma^* \Rightarrow S = \beta\bar{E} - \alpha\bar{E} \in \Gamma^*.$$

Also

$$\gamma(S) = \gamma(\beta\bar{E}) - \gamma(\alpha\bar{E}) = (\beta - \alpha)\gamma(\bar{E}) = (\beta - \alpha)\lambda(E) = \bar{\lambda}(S).$$

Proof of (b):

It suffices to show that for every $f \in L^+$, $\Omega_f \in \bar{\Lambda}^*$ and $\bar{\lambda}(\Omega_f) = \gamma(\Omega_f)$. We first show that $E_a = [f > a] \in \Lambda$ for every a in $0 < a < \infty$.

Setting $f_a = f \cap a$, we have $f_a = a(f/a \cap 1) \in L^+$, and hence also $g_n = n(f - f_a) \in L^+$ and $h_n = g_n \cap 1 \in L^+$ for every n. This implies that $\Omega_{h_n} \in \Gamma$. Also $h_n \uparrow \chi_{Ea}$, so that $\Omega_{h_n} \uparrow \bar{E}_a$. Hence $\bar{E}_a \in \Gamma^*$, and $E_a \in \Lambda$. If now $\{a_n)$ is a countable dense subset of $R_1{}^+$, then $\Omega_f = \bigcup_n (a_n \bar{E}_{a_n})$, and each $a_n \bar{E}_{a_n} \in \bar{\Lambda}$. Hence $\Omega_f \in \bar{\Lambda}^*$ and Ω_f can be expressed as a countable disjoint union of sets in $\bar{\Lambda}$, $\bar{\Lambda}$ being a semi-ring. It will then follow from (a) above that $\gamma(\Omega_f) = \bar{\lambda}(\Omega_f)$.

We have the following theorems as consequences:

Theorem 5 (*cf.* [3], p. 69, Theorem 8)

For the induced measure (Λ, λ), $\Lambda^* = \Lambda$.

Proof

We have only to note that

$$E \in \Lambda^* \Rightarrow \bar{E} \in \bar{\Lambda}^* \Rightarrow \bar{E} \in \Gamma^* \Rightarrow E \in \Lambda.$$

Theorem 6 (*cf.* [3], p. 70, Theorem 9)

The induced measure (Λ, λ) of the elementary integral (L, I) on step functions for a given measure (Θ, θ) coincides with its extended measure space (Θ^*, θ).

Proof

Let (Θ_1, θ_1) be the finite part of (Θ, θ). It is easily seen that $(\Theta^*, \theta) = (\Theta_1^*, \theta_1)$. Also, $(\Lambda, \lambda) = (\Lambda^*, \lambda)$ by theorem 5. We have then to show that $(\Lambda^*, \lambda) = (\Theta_1^*, \theta_1)$.

In turn it suffices to show, by theorem 3, that their associated product measure spaces $(\bar{\Lambda}^*, \bar{\lambda})$ and $(\bar{\Theta}_1^*, \bar{\theta}_1)$ are equal; which, using theorem 4, reduces to showing that $(\Gamma^*, \gamma) = (\bar{\Theta}_1^*, \bar{\theta}_1)$, where (Γ, γ) is the associated product measure of (L, I). In fact, (Γ, γ) and $(\bar{\Theta}_1, \bar{\theta}_1)$ generate the same outer measure: this will follow from lemma 1, if we show that $\gamma^* | \bar{\Theta}_1 = \bar{\theta}_1$ and $\bar{\theta}_1^* | \Gamma = \gamma$; which is very simple.

Remarks

The above theorem shows, that if (L, I) is the elementary integral of step functions for a measure (Λ, λ), then the concepts of measurability, integrability etc., and the values $I(f)$, corresponding to (L, I), coincide with those corresponding to (Λ, λ). Combined with theorem 4, this reconciles completely the classical measure theoretic and the linear functional approaches, under our assumptions.

Department of Mathematics,
Panjab (India) University,
Chandigarh, India

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